Categorification of the colored Jones polynomial and Rasmussen invariant of links

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Abstract: We define a family of formal Khovanov brackets of a colored link depending on two parameters. The isomorphism classes of these brackets are invariants of framed colored links. The Bar-Natan functors applied to these brackets produce Khovanov and Lee homology theories categorifying the colored Jones polynomial. Further, we study conditions under which framed colored link cobordisms induce chain transformations between our formal brackets. We conjecture that for special choice of parameters, Khovanov and Lee homology theories of colored links are functorial (up to sign). Finally, we extend the Rasmussen invariant to links and give examples where this invariant is a stronger obstruction to sliceness than the multivariable Levine–Tristram signature.

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Universal structures in some mean field spin glasses, and an application

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Abstract

We discuss a spin glass reminiscent of the Random Energy Model, which allows in particular to recast the Parisi minimization into a more classical Gibbs variational principle, thereby shedding some light on the physical meaning of the order parameter of the Parisi theory. As an application, we study the impact of an extensive cavity field on Derrida's REM: Despite its simplicity, this model displays some interesting features such as ultrametricity and chaos in temperature.

1 Introduction

After years of intensive research and important advances ([1], [7], [9], [12]), the Parisi theory [10], originally developed in the study of the Sherrington-Kirkpatrick model of spin glasses, still remains mathematically quite elusive.

Despite the spectacular proof by Guerra and Talagrand that Parisi's replica symmetry breaking scheme provides the correct free energy for the SK-model, many aspects of the Parisi ansatz continue to present major challenges. In fact, the appearance of seemingly universal features, such as the Derrida-Ruelle hierarchical structures, the (related) ultrametricity, the law of the pure states, are still far from being understood.

We hope to gain some modest insights into these issues by considering generalizations of the Random Energy Model (REM for short), that is, models with Hamiltonians given by independent random variables. Our generalization is different from the "generalized random energy model" invented by Derrida. It can be analyzed by large deviation techniques. Despite its simplicity, it exhibits a number of interesting properties, like asymptotic ultrametricity, Poisson-Dirichlet description of the pure states, chaos in temperature, and a non-trivial dependence of the overlap structure on the temperature. The free energy is given by a Parisi-type formula which naturally can be linked to a Gibbs variational formula via a kind of duality relation which makes apparent why an infimum appears in the Parisi formulation.

The second part of this work presents a particular mean field spin glass which we call the "REM+Cavity". It is related to the random overlap structures of Aizenman,

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Sims and Starr [1]; but, instead of taking the thermodynamical limit in the REM *first* and *then* perform a one spin perturbation of the Derrida-Ruelle structures, we perform a cavity field perturbation on the finite systems first, and only subsequently do we take the thermodynamical limit. As a first step, we stick here to the simplest finite size counterpart of the Derrida-Ruelle structures, the Random Energy Model [5]. Our model shows a delicate phase transition where Replica Symmetry is broken and ultrametricity sets in. In the low temperature region massive pure states emerge, with law being given by the Poisson-Dirichlet distribution. The model also displays chaotic behavior in temperature. The natural extensions of our approach to models with more intricate dependencies than those of REM-type (such as for instance the Generalized Random Energy Models) turns out to be quite a subtle. We will address this issue in a forthcoming paper.

2 Mean field models of REM-type

Consider a double sequence $X_{\alpha,i}, \alpha, i \geq 1$, of i.i.d. random variables with a distribution μ, taking values in a Polish space $(S, \mathcal{S})$, and which are defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $N \in \mathbb{N}$, $\alpha > 1$, the empirical distributions is defined by

$$L_{N,\alpha} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} \delta_{X_{\alpha,i}},$$

which takes values in $\mathcal{M}_1^+(S)$, the set of probability measures on $(S, \mathcal{S})$, which itself is a Polish space when equipped with the weak topology. Let $\Phi : \mathcal{M}_1^+ \to \mathbb{R}$ be a continuous function. We write

$$Z_N \stackrel{\text{def}}{=} 2^{-N} \sum_{\alpha=1}^{2^N} \exp\left[N\Phi(L_{N,\alpha})\right],\ f_N(\Phi, \mu) \stackrel{\text{def}}{=} \frac{1}{N} \log Z_N.$$

Theorem 2.1
The limit $f(\Phi, \mu) = \lim_{N\to\infty} f_N(\Phi, \mu)$ exists $\mathbb{P} - a.s.$, is non-random, and is given as

$$f(\Phi, \mu) = \sup\left\{\Phi(\nu) - H(\nu \mid \mu) :\ H(\nu \mid \mu) \leq \log 2\right\},$$

where H is the usual relative entropy $H(\nu \mid \mu) \stackrel{\text{def}}{=} \int \log\left(\frac{d\nu}{d\mu}\right) d\nu$ if $\nu \ll \mu$ and $\log(d\nu/d\mu) \in L_1(\mu)$, and $= \infty$ otherwise.

We specialize to linear functionals $\Phi(\nu) = \int \phi d\nu$, $\phi : S \to \mathbb{R}$, i.e.

$$Z_N = 2^{-N} \sum_{\alpha} \exp\left[\sum_{i=1}^{N} \phi(X_{\alpha,i})\right] \tag{2.1}$$

In order that Φ is continuous, we have to assume that ϕ is bounded and continuous, a condition we want to relax somewhat. By a slight abuse of notation, we write $f_N(\phi, \mu)$

for the free energy of the finite-size system, and $f(\phi, \mu)$ for its limit, which by Theorem 2.1 is given through

$$f(\phi, \mu) = \sup \left\{ \int \phi(x)\nu(dx) - H(\nu \mid \mu) : \ H(\nu \mid \mu) \leq \log 2 \right\}, \tag{2.2}$$

at least if ϕ is bounded and continuous. We shall refer to expression (2.2) as the Gibbs variational principle (GVP). Let us write for a distribution $\nu \in \mathcal{M}_1^+(S)$, and $h : S \to \mathbb{R}$, $E_\nu[h] \stackrel{\text{def}}{=} \int h(x)\nu(dx)$, and for $m \in \mathbb{R}$, $\Gamma_\phi(m) \stackrel{\text{def}}{=} \log E_\mu \left[\mathrm{e}^{m\phi}\right]$, which we always assume to exist. We also define the probability measure G_m on S by

$$\frac{dG_m}{d\mu} \stackrel{\text{def}}{=} \frac{\mathrm{e}^{m\phi}}{Z(m)}, \tag{2.3}$$

$Z(m)$ is the appropriate norming constant.

Theorem 2.2
Assume $\phi : S \to \mathbb{R}$ is continuous, and satisfies

$$\int \mathrm{e}^{\lambda\phi} d\mu < \infty \tag{2.4}$$

for all real λ. Then

$$\lim_{N\to\infty} f_N(\phi, \mu) = f(\phi, \mu), \tag{2.5}$$

f given by (2.2). Furthermore, there exists a unique maximizer of the right hand side of (2.2) in the form G_{m_*} where $m_* \in (0, 1]$ is characterized as follows: If

$$\Gamma'_\phi(1) - \Gamma_\phi(1) \leq \log 2, \tag{2.6}$$

then $m_\star = 1$. Otherwise $m_\star \in (0, 1)$ is the unique solution to the following equation:

$$m\Gamma'_\phi(m) - \Gamma_\phi(m) = \log 2. \tag{2.7}$$

If $m_* = 1$, i.e. (2.6) holds, we say the model is in *high temperature*, and otherwise in *low temperature*.

For the Sherrington-Kirpatrick model the free energy was originally obtained by Parisi using the replica method, and a special ansatz for the so-called "replica symmetry breaking". The physical content of Parisi's functional is still somewhat mysterious despite of considerable progress made later. In our setting, the nonrigorous RSB-mechanism would yield the following free energy for a spin glass of the form (2.1) as

$$\mathrm{Parisi}(\phi, \mu) \stackrel{\text{def}}{=} \inf_{m\in[0,1]} \left\{ \frac{\log 2}{m} + \frac{1}{m} \log E_\mu \mathrm{e}^{m\phi} - \log 2 \right\}, \tag{2.8}$$

The fact that one takes the infimum instead of the usual supremum in the Gibbs formalism is at first sight rather puzzling. However, in our setup, the identification of (2.2) with the right-hand side of (2.8) will be rather straightforward, and we have

Theorem 2.3

$$f(\phi,\mu) = \text{Parisi}\,(\phi,\mu)$$

We learned from Guerra [8] a simple argument how to prove that $f(\phi,\mu)$ is bounded by (2.8): For $m \in [0,1]$,

$$\begin{aligned} f_N(\phi,\mu) &= \frac{1}{N}\log\left(2^{-N}\sum_\alpha \exp\left[\sum\nolimits_i \phi(X_{\alpha,i})\right]\right) \\ &= \frac{1}{mN}\log\left(2^{-N}\sum_\alpha \exp\left[\sum\nolimits_i \phi(X_{\alpha,i})\right]\right)^m \\ &\le \frac{1}{mN}\log\left(2^{-mN}\sum_\alpha \exp\left[m\sum\nolimits_i \phi(X_{\alpha,i})\right]\right), \end{aligned}$$

where the last bound follows by straightforward convexity/concavity arguments. Taking expectation w.r.t. the randomness, exploiting concavity of the logarithm, the independence of the random variables appearing in the sum $\sum_i \phi(X_{\alpha,i})$, and optimizing over $m \in [0,1]$ one easily gets

$$\mathbb{E}f_N(\phi,\mu) \le \text{Parisi}\,(\phi,\mu)\,, \tag{2.9}$$

uniformly in N. We will not use that, and we will give another proof of Theorem 2.3 in Section 4.1.

As usual, the Gibbs measure is defined by

$$\mathcal{G}_{\Phi,N}(\alpha) \stackrel{\text{def}}{=} \frac{2^{-N}\exp\left[N\Phi(L_{N,\alpha})\right]}{Z_N},\ 1 \le \alpha \le 2^N.$$

We analyze this only in the linear case $\Phi(\nu) = \int \phi d\nu$. By an abuse of notation, we write it simply as $\mathcal{G}_{\phi,N}(\alpha)$.

We recall the definition of the Poisson-Dirichlet point process with parameter $m \in (0,1)$. We first consider a Poisson point process on $\mathbb{R}^+$ with intensity measure $t^{-m-1}dt$. We call such a point process a $\text{PP}\,(m)$. This point process has countably many single points with a maximal element. If we order the points downwards, we obtain a sequence of random variables $\xi_1 > \xi_2 > \cdots$. If $m < 1$, then $\zeta \stackrel{\text{def}}{=} \sum_i \xi_i < \infty$, almost surely, and we can define $\overline{\xi}_i \stackrel{\text{def}}{=} \xi_i/\zeta$. Then $\sum_i \delta_{\overline{\xi}_i}$ is a Poisson-Dirichlet point process with parameter m. We write $\text{PD}\,(m)$ for such a point process.

Theorem 2.4
Suppose that ϕ,μ are such that the system is in low temperature, i.e. $m_* < 1$. (m_* the unique solution to the entropy condition (2.7)). Assume furthermore that the distribution of ϕ under μ is non-lattice. Then the point process $\sum_\alpha \delta_{\mathcal{G}_{\phi,N}(\alpha)}$ converges weakly as $N \to \infty$ to a $\text{PD}\,(m_\star)$.

Remark that Theorem 2.4 accounts for some universality of the Derrida-Ruelle structures and the so-called Poisson-Dirichlet distribution, which naturally arise in the weak limits of the Gibbs measure associated to a REM-system in low temperature.

3 The REM+Cavity model

We give an application of the previous results. Let again $N \in \mathbb{N}$. We set $\Sigma_N \stackrel{\text{def}}{=} \{1, \dots, 2^N\}$ and consider on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ a sequence $(X_\alpha, \alpha \in \Sigma_N)$ of independent, centered Gaussians with variance N, as well as another independent sequence $(g_{\alpha,i}, \alpha \in \Sigma_N, i = 1, \dots, N)$ of independent standard Gaussians. For $\alpha \in \Sigma_N, \sigma = (\sigma_1, \dots, \sigma_N) \in \{\pm 1\}^N$, we define the Hamiltonian of the REM+Cavity-model.

$$H(\alpha,\sigma) \stackrel{\text{def}}{=} X_\alpha + \sum_{i=1}^N g_{\alpha,i}\sigma_i. \tag{3.1}$$

$H(\cdot,\cdot)$ is thus a Gaussian field on $\Sigma_N \times \{\pm 1\}^N$ with covariance given by

$$\mathbb{E}[H(\alpha,\sigma)H(\alpha,\sigma')] = N\delta_{\alpha=\alpha'} + N\delta_{\alpha=\alpha'}q(\sigma,\sigma'),$$

where $q(\sigma,\sigma') \stackrel{\text{def}}{=} \frac{1}{N}\sum_{i=1}^N \sigma_i\sigma_i'$ is the usual overlap of the configurations σ, σ'. For $\beta \in \mathbb{R}$, the inverse of the temperature, we define the free energy

$$f_N(\beta) \stackrel{\text{def}}{=} \frac{1}{N}\log\left[2^{-2N}\sum_{\alpha,\sigma}\exp\left(\beta H(\alpha,\sigma)\right)\right]. \tag{3.2}$$

Proposition 3.1
The limit $f(\beta) = \lim_{N\to\infty} f_N(\beta)$ exists $\mathbb{P}$-a.s. and is given by

$$f(\beta) = \begin{cases} \beta^2 & \text{if } \beta \le \beta_{\text{cr}} \\ \frac{\beta^2}{2}m_\star + \frac{E[\cosh(\beta g)^{m_\star}\log\cosh(\beta g)]}{E[\cosh(\beta g)^{m_\star}]} - \log 2 & \text{if } \beta > \beta_{\text{cr}} \end{cases} \tag{3.3}$$

with $\beta_{\text{cr}} > 0$ being the unique positive solutions of the equation

$$E[\cosh(\beta g)\log\cosh(\beta g)] = \mathrm{e}^{\beta^2/2}\log 2, \tag{3.4}$$

and for $\beta > \beta_{\text{cr}}$, $m_\star = m_\star(\beta) \in (0,1)$ is the unique solution of

$$\frac{\beta^2}{2}m^2 - \log E[\cosh(\beta g)^m] + m\frac{E[\cosh(\beta g)^m\log\cosh(\beta g)]}{E[\cosh(\beta g)^m]} = \log 2. \tag{3.5}$$

The mechanism behind this formula is easy to understand. Remark first that we can write the partition function as

$$\begin{aligned} 2^{-2N}\sum_{\alpha,\sigma}\exp\left(\beta H(\alpha,\sigma)\right) &= 2^{-N}\sum_\alpha \mathrm{e}^{\beta X_\alpha}2^{-N}\sum_\sigma \exp\left[\beta\sum\nolimits_{i=1}^N g_{\alpha,i}\sigma_i\right] \\ &= 2^{-N}\sum_\alpha \mathrm{e}^{\beta X_\alpha}\prod_{i=1}^N\cosh\left(\beta g_{\alpha,i}\right) \\ &= 2^{-N}\sum_\alpha\exp\left[\beta X_\alpha + \sum\nolimits_{i=1}^N\log\cosh\left(\beta g_{\alpha,i}\right)\right] \\ &= 2^{-N}\sum_\alpha\exp\left[\beta X_\alpha + N\int\log\cosh\left(\beta\right)L_{N,\alpha}\left(dx\right)\right], \end{aligned}$$

where

$$L_{N,\alpha} \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} \delta_{g_{\alpha,i}}.$$

The probability that for a fixed α, we have $X_\alpha \approx yN$, and $L_{N,\alpha} \approx \nu$

$$\approx \exp\left[-Ny^2/2 - NH\left(\nu \mid \mu\right)\right],$$

μ being here the standard normal distribution. Arguing roughly in the same way as before, we conclude that

$$f(\beta) = \sup_{y,\nu} \Big\{ \beta y + \int \log\cosh\left(\beta y\right) \nu\left(dy\right) - y^2/2 - H\left(\nu \mid \mu\right) \qquad (3.6)$$
$$: y^2/2 + H\left(\nu \mid \mu\right) \leq \log 2 \Big\},$$

which leads to the expression in Proposition 3.1. β_{cr} is the value for which the restriction $y^2/2 + H\left(\nu \mid \mu\right) \leq \log 2$ becomes relevant in the supremum. An interesting feature is that for any β, the α's which are giving the main contribution to the partition function are those for which

$$L_{N,\alpha} \approx \nu,$$

ν being the maximizer in the variational problem. We will give a precise derivation in Section 4.4.

According to the convention following Theorem 2.2, we call the region $\beta \leq \beta_{\text{cr}}$ the high-temperature, and $\beta > \beta_{\text{cr}}$ the low temperature regime. The associated Gibbs measure is:

$$\mathcal{G}_{\beta,N}(\alpha,\sigma) = \frac{\exp \beta H(\alpha,\sigma)}{\sum_{\alpha',\sigma'} \exp \beta H(\alpha',\sigma')}, \text{ for}(\alpha,\sigma) \in \Sigma_N \times \{\pm 1\}^N.$$

It is not difficult to realize that even in low temperature, the Gibbs weights of individual configurations are exponentially small in N. To get a macroscopic weight we must lump together exponentially many configurations. In the present situation, we have to take the marginal measure on the first component:

$$\mathcal{G}^{(1)}_{\beta,N}(\alpha) \stackrel{\text{def}}{=} \sum_{\sigma\in\{\pm1\}^N} \mathcal{G}_{\beta,N}(\alpha,\sigma).$$

Proposition 3.2
If $\beta > \beta_{\text{cr}}$, then the point process $\sum_\alpha \delta_{\mathcal{G}^{(1)}_{\beta,N}(\alpha)}$ converges weakly to $\mathrm{PD}\left(m_*\right)$.

We thus witness in the low-temperature regime of the REM+Cavity the emergence of massive pure states, with law being given by the Poisson-Dirichlet distribution.

We can also derive the limiting behavior of the overlaps under the replicated Gibbs measure $\mathcal{G}^{\otimes 2}_{\beta,N}$. Following the physicists convention, we write $\langle\cdot\rangle^{\otimes 2}_{\beta,N}$ for the expectation with respect to $\mathcal{G}^{\otimes 2}_{\beta,N}$. From the above proposition, it is clear that in the $N \to \infty$

limit, $\mathcal{G}_{\beta,N}^{\otimes 2}(\alpha=\alpha')$ has the same distribution as $\sum \eta_i^2$, where the η_i are the points of a PD (m_*). Here α,α' are the first components of the two replicas. The expectation of $\sum \eta_i^2$ is well known to be $1-m_*$. Therefore, we get

$$\lim_{N\to\infty} \mathbb{E}\mathcal{G}_{\beta,N}^{\otimes 2}\left(\alpha=\alpha'\right)=1-m_*. \tag{3.7}$$

Conditioned on $\alpha\neq\alpha'$, the overlap of σ,σ' is 0, in the $N\to\infty$ limit, whereas conditioned on $\alpha=\alpha'$, it is given by

$$q_\star \stackrel{\text{def}}{=} \frac{E\left[\tanh^2(\beta g)\exp\left(m_\star \log\cosh(\beta g)\right)\right]}{E\left[\exp(m_\star \log\cosh(\beta g))\right]}, \tag{3.8}$$

for g a standard Gaussian and E denoting expectation with respect to it. To phrase it precisely

Proposition 3.3 (Ultrametricity for the REM+Cavity)
For $\beta>\beta_{\text{cr}}$,

$$\lim_{N\to\infty}\mathbb{E}\left[\left\langle \delta_{\alpha=\alpha'}\left(q(\sigma,\sigma')-q_\star\right)^2\right\rangle_{\beta,N}^{\otimes 2}\right]=0, \tag{3.9}$$

and

$$\lim_{N\to\infty}\mathbb{E}\left[\left\langle \delta_{\alpha\neq\alpha'} q\left(\sigma,\sigma'\right)^2\right\rangle_{\beta,N}^{\otimes 2}\right]=0. \tag{3.10}$$

It should be remarked that our REM+Cavity model is *not* ultrametrically structured for finite N, in contrast to the pure REM or the GREM. This means that the natural L_2-metric on the Gaussian Hamiltonians is *not* an ultrametric on $\Sigma_N\times\{-1,1\}^N$.

An interesting feature of our model is that it exhibits the so-called "chaos in temperature", in sharp contrast with the pure REM which does not have this property. The effect is easy to understand. For a temperature parameter $\beta>\beta_{\text{cr}}$, $\mathcal{G}_{\beta,N}^{(1)}$ picks from the α for which $X_\alpha\approx y_\beta^*$, $L_{N,\alpha}\approx\nu_\beta^*$, $\left(y_\beta^*,\nu_\beta^*\right)$ being the maximizer of (3.6). y_β^*,ν_β^* depend in a non-trivial way on β. In particular, they change when β is changed, regardless how large β is. Therefore the contribution to the partition function is coming from a completely different set of α's if one changes the temperature parameter. This is in contrast to the REM where for β above the critical parameter, the α's which contribute are always those for which the X_α are close to the maximal possible value.

To phrase the property precisely, we have the following result:

Proposition 3.4 (Chaos in temperature for the REM+Cavity)
Assume $\beta,\beta'>\beta_\star$ and $\beta\neq\beta'$. There exists $\delta>0$ such that

$$\mathbb{P}\left[\mathcal{G}_{\beta,N}\otimes\mathcal{G}_{\beta',N}(\delta_{\alpha=\alpha'})\geq \mathrm{e}^{-\delta N}\right]\leq \mathrm{e}^{-\delta N}, \tag{3.11}$$

and

$$\lim_{N\to\infty}\mathbb{E}\left\langle \delta_{\alpha\neq\alpha'} q\left(\sigma,\sigma'\right)^2\right\rangle_{\beta,\beta',N}^{\otimes 2}=0. \tag{3.12}$$

Summarizing, we have the following situation for the $N \to \infty$ the Gibbs measure at $\beta > \beta_{\mathrm{cr}}$: It gives macroscopic weights to α's for which $L_{N,\alpha}$ is approximately ν^*_β. The random weights are given by a Poisson-Dirichlet point process with parameter m_*. If in the replicated system, $\alpha \neq \alpha'$, then the corresponding σ, σ' have zero overlap with probability ≈ 1. On the other hand, if $\alpha = \alpha'$, then also the σ, σ' have a non-zero overlap, given by q_*. If β changes, then the choice is made from a completely different group of α's.

4 Proofs of the main results

4.1 The free energy for spin glasses of REM-type

For the proof of Theorem 2.1, a technical result is needed. Given $A \subset \mathcal{M}_1^+(S)$, we set

$$M_N(A) \stackrel{\mathrm{def}}{=} \#\left\{\alpha \leq 2^N : L_{N,\alpha} \in A\right\}.$$

We also write $H(A)$ for $\inf_{\nu \in A} H(\nu \mid \mu)$.

Lemma 4.1
Let $\nu \in \mathcal{M}_1^+(S)$, and V be an open neighborhood of ν. If $H(\nu \mid \mu) \leq \log 2$, and $\varepsilon > 0$, then there exists an open neighborhood U of ν, $U \subset V$, and $\delta > 0$ such that for large enough N

$$\mathbb{P}\Big[M_N(U) \leq \exp\left[N(\log 2 - H(\nu \mid \mu) - \varepsilon)\right]\Big] \leq \mathrm{e}^{-N\delta}, \tag{4.1}$$

$$\mathbb{P}\Big[M_N(U) \geq \exp\left[N(\log 2 - H(\nu \mid \mu) + \varepsilon)\right]\Big] \leq \mathrm{e}^{-N\delta}. \tag{4.2}$$

If $H(\nu) > \log 2$, then there exist U and δ as above, with

$$\mathbb{P}\Big[M_N(U) \neq 0\Big] \leq \mathrm{e}^{-N\delta}.$$

Proof. Let first $\nu \in \mathcal{M}_1^+(S)$ satisfy $H(\nu \mid \mu) \leq \log 2$. The statement of the Lemma is trivial if $\nu = \mu$, so we assume $\nu \neq \mu$. Let $B_r(\nu) \subset \mathcal{M}_1^+(S)$ be the open ball of radius r and center ν, where we have equipped $\mathcal{M}_1^+(S)$ with one of the standard metrics, e.g. Prohorov's metric. Then $H(B_r(\nu)) = H(\mathrm{cl}(B_r(\nu)))$, except for countably many r. Therefore we can find arbitrary small $\varepsilon_1 > 0$, and $U \stackrel{\mathrm{def}}{=} B_r(\nu) \subset V$, such that $H(U) = H(\mathrm{cl}(U)) = H(\nu \mid \mu) - \varepsilon_1$. From Sanov's Theorem, we have

$$\mathbb{P}\left(L_{N,\alpha} \in U\right) \geq \exp\left[-N\left(H(\nu \mid \mu) - \frac{5}{6}\varepsilon_1\right)\right],$$

$$\mathbb{P}\left(L_{N,\alpha} \in \mathrm{cl}(U)\right) \leq \exp\left[-N\left(H(\nu \mid \mu) - \frac{7}{6}\varepsilon_1\right)\right]$$

for large enough N. Therefore

$$\mathbb{E}M_N(U) \geq \exp\left[N\left(\log 2 - H(\nu \mid \mu) + \frac{5}{6}\varepsilon_1\right)\right],$$

$$\mathbb{E}M_N(U) \leq \mathbb{E}M_N(\mathrm{cl}\, U) \leq \exp\left[N\left(\log 2 - H(\nu \mid \mu) + \frac{7}{6}\varepsilon_1\right)\right],$$

and using the independence of the $L_{N,\alpha}$

$$\mathbb{E}M_N(U)^2 \leq (\mathbb{E}M_N(U))^2 + \exp\left[N\left(\log 2 - H(\nu \mid \mu) + \frac{7}{6}\varepsilon_1\right)\right],$$

$$\mathrm{var}_{\mathbb{P}}\, M_N(U) \leq \mathrm{e}^{-N\varepsilon_1/2} \left(\mathbb{E}M_N(U)\right)^2 \tag{4.3}$$

Hence,

$$\mathbb{P}\Big[M_N(U) \leq \Big(1 - \mathrm{e}^{-N\varepsilon_1/8}\Big)\, \mathbb{E}M_N(U)\Big] \leq \exp\left[-N\varepsilon_1/4\right],$$

$$\mathbb{P}\Big[M_N(U) \geq \Big(1 + \mathrm{e}^{-N\varepsilon_1/8}\Big)\, \mathbb{E}M_N(U)\Big] \leq \exp\left[-N\varepsilon_1/4\right].$$

Choosing ε_1 smaller than $\varepsilon/2$ and $\delta = \varepsilon_1/4$ proves the Lemma in this case. The case $H(\nu \mid \mu) > \log 2$ needs only a slight modification. In that case, there exists an open neighborhood $U \subset V$ of ν such that $\mathbb{P}\left(L_{N,\alpha} \in U\right)$ is exponentially small in N, with a decay rate which is bigger than $\log 2$. This proves the claim by the Markov inequality. ■

Proof of Theorem 2.1. We first prove the lower bound. Let ν be any element in $\mathcal{M}_1^+(S)$ satisfying $H(\nu \mid \mu) \leq \log 2$. Let $\varepsilon > 0$. As Φ is continuous, we can choose an open neighborhood V of ν satisfying $\mid \Phi(\gamma) - \Phi(\nu) \mid \leq \varepsilon$ for $\gamma \in V$. Applying Lemma 4.1 we find a neighborhood U of ν in V satisfying (4.1). As

$$Z_N \geq 2^{-N} \exp\left[N \inf_{\gamma \in U} \Phi(\gamma)\right] M_N(U) \geq 2^{-N} \mathrm{e}^{N(\Phi(\gamma)-\varepsilon)} M_N(U),$$

we get from Lemma 4.1 that $\mathbb{P}$-a.s. one has eventually

$$\begin{aligned} Z_N &\geq 2^{-N} \exp\left[N\left(\Phi(\gamma) - \varepsilon\right)\right] \exp\left[N\left(\log 2 - H(\nu \mid \mu) - \varepsilon\right)\right] \\ &\geq \exp\left[N\left\{\Phi(\nu) - H(\nu \mid \mu) - 2\varepsilon\right\}\right], \end{aligned}$$

and therefore

$$\liminf_{N\to\infty} \frac{1}{N} \log Z_N \geq \Phi(\nu) - H(\nu \mid \mu)$$

almost surely, for all ν with $H(\nu \mid \mu) \leq \log 2$. This proves the lower bound.

We now prove the upper bound. We use the well-known fact that there exists a compact set $K \subset \mathcal{M}_1^+(S)$ such that $\mathbb{P}\left(L_N \notin K\right) \leq \exp\left[-N(\log 2 + 1)\right]$. Let D_N be the event

$$D_N \stackrel{\mathrm{def}}{=} \bigcap_{\alpha=1}^{2^N} \left\{L_{N,\alpha} \in K\right\}.$$

Then

$$\mathbb{P}\left[D_N^c\right] \leq 2^N 2\mathrm{e}^{-N(\log 2+1)},$$

and therefore

$$\mathbb{P}\left[\liminf_{N\to\infty} D_N\right] = 1.$$

Fix $\varepsilon > 0$. For any $\nu \in K$, we choose V_ν such that $\Phi(\gamma) - \Phi(\nu) \leq \varepsilon$ for $\gamma \in V_\nu$, and then $U_\nu \subset V_\nu$ according to Lemma 4.1. The U_ν cover K, and we choose a finite subcover, call it $U_{\nu_1}, \ldots, U_{\nu_m}$. Then, on $D \stackrel{\text{def}}{=} \liminf_N D_N$ we have, writing U_k instead of U_{ν_k},

$$\begin{aligned}
Z_N &= 2^{-N} \sum_\alpha \exp\left[N\Phi(L_{N,\alpha})\right] \\
&= 2^{-N} \sum_{k=1}^m \sum_{\alpha : L_{N,\alpha} \in U_k} \exp\left[N\Phi(L_{N,\alpha})\right] \\
&\leq 2^{-N} \sum_{k=1}^m \exp\left[N\left\{\Phi(\nu_k) + \varepsilon\right\}\right] M_N(U) \\
&\leq \sum_{k : H(\nu_k|\mu) \leq \log 2} \exp\left[N\left\{\Phi(\nu_k) - H(\nu_k \mid \mu) + 2\varepsilon\right\}\right]
\end{aligned}$$

outside an event which has probability at most $m \exp\left[-N \min_{j \leq m} \delta_j\right]$, where the δ_j corresponds to the U_j. From this estimate on gets that $\mathbb{P}$-a.s. one has

$$\limsup_{N\to\infty} \frac{1}{N} \log Z_N \leq \sup_{\nu : H(\nu|\mu) \leq \log 2} \left[\Phi(\nu) - H(\nu \mid \mu)\right],$$

which together with the lower bound settles the proof of Theorem 2.1. ■

Proof of Theorem 2.2. To prove (2.5), we cannot directly apply Theorem 2.1 unless ϕ is bounded. We therefore truncate ϕ by defining $\phi_M(x) \stackrel{\text{def}}{=} \min\left(M, \max\left(\phi(x), -M\right)\right)$, $M > 0$, which is bounded and continuous. As a consequence of our assumption (2.4), we have that for any $\varepsilon > 0$, and $K > 0$, we can find M such that

$$\mathbb{P}\left(\left|\sum\nolimits_i \left(\phi\left(X_{\alpha,i}\right) - \phi_M\left(X_{\alpha,i}\right)\right)\right| \geq \varepsilon N\right) \leq \exp\left[-KN\right].$$

If we choose $K > \log 2$, then with probability going to 0 exponentially fast in N, there is no α such that $\left|\sum_i \left(\phi\left(X_{\alpha,i}\right) - \phi_M\left(X_{\alpha,i}\right)\right)\right| \geq \varepsilon N$. In particular

$$\begin{aligned}
\sum_\alpha \exp\left[-\sum\nolimits_i \phi_M\left(X_{\alpha,i}\right) - \varepsilon N\right] &\leq \sum_\alpha \exp\left[-\sum\nolimits_i \phi\left(X_{\alpha,i}\right)\right] \\
&\leq \sum_\alpha \exp\left[-\sum\nolimits_i \phi_M\left(X_{\alpha,i}\right) + \varepsilon N\right].
\end{aligned}$$

Applying Theorem 2.1 to ϕ_M, and passing to the $M \to \infty$ limit in the end, proves (2.5).

The more complicated claim is the one on the characterization of the maximizer.

We first restrict the analysis of the GVP (2.2) to measures G_m of the form (2.3). Restricting to the variational formula to these measures evidently yields a lower bound to the GVP, which actually reads

$$\sup_{m\in\mathbb{R}} \left\{(1-m)\Gamma'_\phi(m) + \Gamma_\phi(m) : m\Gamma'_\phi(m) - \Gamma_\phi(m) \leq \log 2\right\}. \tag{4.4}$$

We now claim that the target function $(1-m)\Gamma'_\phi(m)+\Gamma_\phi(m)$ is increasing on $m \in (-\infty, 1]$ and decreasing otherwise; in fact

$$\frac{d}{dm}\left[(1-m)\Gamma'_\phi(m)+\Gamma_\phi(m)\right] = (1-m)\Gamma''_\phi(m)$$

and $\Gamma''_\phi(m) > 0$, $\forall m \in \mathbb{R}$. Thus, we can restrict the search for a maximizing $m \in \mathbb{R}$ in (4.4) to:

$$\sup_{m\in(-\infty,1]} \left\{(1-m)\Gamma'_\phi(m)+\Gamma_\phi(m) : m\Gamma'_\phi(m)-\Gamma_\phi(m) \le \log 2\right\} \tag{4.5}$$

But monotonicity also implies that the (global) maximum is attained in $m = 1$, i.e. equals $\Gamma_\phi(1)$ as long as the side condition $\Gamma'_\phi(1)-\Gamma_\phi(1) \le \log 2$, i.e. we are in the high temperature region.

In the low temperature region, i.e. if $\Gamma'_\phi(1)-\Gamma_\phi(1) > \log 2$, we first observe that the function $m \to m\Gamma'_\phi(m)-\Gamma_\phi(m)$ is also increasing, this time for any value of $m \ge 0$:

$$\frac{d}{dm}\Big(m\Gamma'_\phi(m)-\Gamma_\phi(m)\Big) = m\Gamma''_\phi(m).$$

Hence, monotonicity of both target and constraint function yields that the maximum is achieved at the largest possible value, which is the one satisfying:

$$m\Gamma'_\phi(m)-\Gamma_\phi(m) = \log 2. \tag{4.6}$$

As the left-hand side is $0 < \log 2$ at $m = 0$, and $> \log 2$ at $m = 1$, continuity and strict monotonicity implies that there is a unique $m_\star \in (0,1)$ satisfying this equation.

It remains to prove that any maximizer of (2.2) is G_{m_*}. For an arbitrary probability measure ν on S, we set

$$\psi(\nu) \stackrel{\text{def}}{=} E_\nu(\phi) - H(\nu \mid \mu).$$

We compute the entropy of ν relative to G_m:

$$\begin{aligned}
H(\nu \mid G_m) &= E_\nu \log \frac{d\nu}{dG_m} = E_\nu \log\left(\frac{d\nu}{d\mu}\cdot\frac{d\mu}{dG_m}\right) \\
&= H(\nu \mid \mu) - E_\nu \log \frac{dG_m}{d\mu} \\
&= H(\nu \mid \mu) - mE_\nu\phi + \log Z(m) \\
&= H(\nu \mid \mu) - mE_\nu\phi + mE_{G_m}\phi - H(G_m \mid \mu)
\end{aligned}$$

the last equality stemming from the definition of the G_m, according to which

$$H(G_m \mid \mu) = mE_{G_m}[\phi] - \log Z(m)$$

An elementary computation yields

$$m\left(\psi(G_m) - \psi(\nu)\right) = H(\nu \mid G_m) + (1-m)\left[H(G_m \mid \mu) - H(\nu \mid \mu)\right].$$

This is true for any m, but we apply in now to $m = m_*$. Then either $1 - m_* = 0$, or $H(G_{m_*} \mid \mu) = \log 2$. Therefore, if $H(\nu \mid \mu) \leq \log 2$, then the right hand side above is non-negative, implying that $\psi(\nu) \leq \psi(G_{m_*})$, and equality only if $H(\nu \mid G_{m_*}) = 0$, i.e. $\nu = G_{m_*}$. This proves the claim. ■

Proof of Theorem 2.3. To prove the equivalence of Parisi and Gibbs Variational Principles, we consider the function $\Psi(m) \stackrel{\text{def}}{=} \frac{\log 2}{m} + \frac{1}{m} \log E_\mu \left[\mathrm{e}^{m\phi} \right] - \log 2$. Recall that Parisi variational principle amounts to minimize $\Psi(m)$ over $[0,1]$. Clearly, either is the minimum attained on the boundary value $m = 1$, or in m solution of $\Psi'(m) = 0$. In case the optimal value is attained in $m = 1$, one immediately sees that $\inf_{m \in [0,1]} \Psi(m) = \Psi(1) = \Gamma_\phi(1)$, thus exactly as in the high-temperature case of Theorem 2.2. Otherwise, it is crucial to remark that

$$\Psi'(m) = \frac{1}{m^2} \left\{ H(G_m \mid \mu) - \log 2 \right\}.$$

Therefore, $\Psi'(m) = 0$ iff $H(G_m \mid \mu) = \log 2$. By Theorem 2.2, we already know that the generalized Gibbs measure associated to the solution of the latter equation is optimal. It is also a simple algebraic exercise to check that with $m_\star$ such that $\Psi'(m_\star) = 0$ one also has $\Psi(m_\star) = \Gamma'_\phi(m_\star) - \log 2$, showing the equivalence of Parisi and Gibbs variational principle in the low temperature case as well. ■

Remark 4.2
The above considerations also show that the order parameter of the Parisi formulation, the optimal $m_\star$ (with either $m_\star = 1$ or such that $\Psi(m_\star) = 0$), is in fact the inverse of temperature of the generalized Gibbs measure solving the Gibbs variational principle. Moreover, derivatives of the target function Ψ in the Parisi formulation naturally appear in terms of entropies of the generalized Gibbs measures relative to the underlying random media.

4.2 The Gibbs measure of spin glasses of REM-type

Let $m_\star$ be as defined in Theorem 2.2, and $G = G_{m_\star}$ the associated measure. We write $v^2 := \mathrm{var}_G(\phi)$ for the variance of ϕ under G. Then define

$$a_N \stackrel{\text{def}}{=} E_G(\phi) N + \omega(N), \tag{4.7}$$

where $\omega(N) \stackrel{\text{def}}{=} -\frac{1}{m_\star} \log \sqrt{2\pi v^2 N}$. For $\alpha \in \{1, \ldots, 2^N\}$ let us also abbreviate $H_N(\alpha) \stackrel{\text{def}}{=} \sum_{i=1}^N \phi(X_{\alpha,i})$.

We begin with a technical result:

Lemma 4.3
Assume that the measure $\mu\phi^{-1}$ on $\mathbb{R}$ is non-lattice. Then, with the above notations, for any $t \in \mathbb{R}$

$$\lim_{N \to \infty} 2^N \mathbb{P} \left[\sum_{i=1}^N \phi(X_{1,i}) - a_N \geq t \right] = \frac{1}{m_\star} \mathrm{e}^{-m_\star t}. \tag{4.8}$$

Proof. We use the usual transformation of measure argument, writing the probability on the left-hand side of (4.8) in terms of a new sequence $\left\{\widetilde{X}_i\right\}$ of independent random variables with distribution function G. As G is equivalent to μ, also $\phi\left(\widetilde{X}_i\right)$ is non-lattice. We write G_N for the distribution of

$$\sum_{i=1}^{N}\left(\phi(\widetilde{X}_i)-E_G(\phi)\right),$$

and $\hat{G}_N$ for the standardized one: $\hat{G}_N(\cdot)=G_N\left(v\sqrt{N}\cdot\right)$.

By change of measure and integration by parts, it holds:

$$\mathbb{P}\left[\sum\nolimits_{i=1}^{N}\phi(X_{1,i})-a_N\geq t\right]=\exp N\left[\log E[\mathrm{e}^{m_\star\phi}]-m_\star E_G(\phi)\right]\int_{t+\omega(N)}\mathrm{e}^{-m_\star y}G_N(dy)$$

Recall that in low temperature,

$$\log E\left(\mathrm{e}^{m_*\phi}\right)-m_*E_G(\phi)=-\log 2,$$

and therefore

$$\begin{aligned}2^N\mathbb{P}\left[\sum\nolimits_{i=1}^{N}\phi(X_{1,i})-a_N\geq t\right]&=\int_{t+\omega(N)}\mathrm{e}^{-m_\star y}G_N(dy)\\&=\int_{t+\omega(N)}\int_{m_\star y}^{\infty}\mathrm{e}^{-z}dzG_N(dy)\\&=\int_{m_*(t+\omega(N))}^{\infty}dz\mathrm{e}^{-z}\hat{G}_N\left(\left(\frac{t+\omega(N)}{\sqrt{N}v},\frac{z}{\sqrt{N}vm_*}\right]\right).\end{aligned}$$

We use now the assumption that $\phi(\widetilde{X}_i)$ has a non-lattice distribution. This implies that we can approximate the distribution function of $\hat{G}_N$ by a smooth distribution function, up to an error of order $o\left(N^{-1/2}\right)$. More precisely, if we define GaussDF to be the distribution function of the standard normal distribution, and φ the density, then, uniformly in x,

$$\hat{G}_N((-\infty,x])=\mathrm{GaussDF}(x)-\frac{\mu_3}{6v^3\sqrt{N}}\left(x^2-1\right)\varphi(x)+o\left(N^{-1/2}\right),\tag{4.9}$$

where μ_3 is the third moment of $\phi(\widetilde{X}_i)-E_G(\phi)$. (see e.g. [6], Theorem XVI.4.1). If we replace $\hat{G}_N$ above by the standard normal distribution, and transform back, we get an expression

$$\exp\left(\frac{m_\star^2}{2}Nv\right)\int_t^{\infty}\exp\left[-\frac{1}{2Nv}\left(z-m_\star Nv+\omega(N)\right)^2\right]\frac{dz}{\sqrt{2\pi Nv}}=\int_t^{\infty}\mathrm{e}^{-m_\star y+o(1)}dy.$$

In order to prove the lemma, it therefore suffices to prove that the two error summands in (4.9) contribute nothing in the $N \to \infty$ limit. This is evident for $o\left(N^{-1/2}\right)$ as

$$\int_{m_*(t+\omega(N))}^{\infty} dz \mathrm{e}^{-z} = O\left(\sqrt{N}\right).$$

For the middle Edgeworth term in (4.9), the special form is of no importance, and we only use that it is of the form $h(x)/\sqrt{N}$ with a bounded smooth function h :

$$\int_{m_*(t+\omega(N))}^{\infty} dz \mathrm{e}^{-z} \left[h\left(\frac{z}{\sqrt{N} v m_*}\right) - h\left(\frac{t+\omega(N)}{\sqrt{N} v}\right)\right] = O(1).$$

■

Proposition 4.4
Within the above setting:

a) *The point process* $\sum_\alpha \delta_{H_N(\alpha)-a_N}$ *converges weakly to a Poisson point process with intensity measure* $\mathrm{e}^{-m_* t} dt$.

b) *The point process* $\sum_\alpha \delta_{\exp[H_N(\alpha)-a_N]}$ *converges weakly to a* $\mathrm{PP}(m_*)$.

Proof. b) is evident from a). For the first claim, we shall exploit the equivalence of weak convergence and convergence of Laplace functionals. For a continuous function with compact support $F \in C_o(\mathbb{R})$ we have:

$$\begin{aligned} \mathbb{E}\mathrm{e}^{-\sum\limits_\alpha F(H_N(\alpha)-a_N)} &= \left\{\mathbb{E}\mathrm{e}^{-F(H_N(1)-a_N)}\right\}^{2^N} \\ &= \left\{1 - \mathbb{E}\left[1 - \mathrm{e}^{-F(H_N(1)-a_N)}\right]\right\}^{2^N}. \end{aligned}$$

By Lemma 4.3, this converges to $\exp\left[-\int\left(1-\mathrm{e}^{-F(t)}\right)\mathrm{e}^{-m_* t} dt\right]$, settling a). ■

Since

$$\mathcal{G}_{N,\phi}(\alpha) = \frac{\exp\left[H_N(\alpha) - a_N\right]}{\sum_{\alpha'} \exp\left[H_N(\alpha') - a_N\right]},$$

to prove Theorem 2.4 it suffices to prove that in low temperature the normalization commutes with taking the $N \to \infty$ limit. For this, we have the following:

Lemma 4.5
Suppose ϕ *is such that the system is in low temperature, and let* $\varepsilon > 0$. *There exists* $C > 0$ *such that*

$$\mathbb{P}\left[\sum\nolimits_\alpha \exp\left[H_N(\alpha) - a_N\right] 1_{H_N(\alpha)-a_N \notin [-C,C]} \geq \varepsilon\right] \leq \varepsilon,$$

for large enough N.

Proof. First, by Lemma 4.3 we clearly have that

$$\mathbb{P}\left[\sum_\alpha \mathrm{e}^{H_N(\alpha)-a_N} 1_{H_N(\alpha)-a_N\geq C} \geq \varepsilon\right] \leq \mathbb{P}\left[\exists \alpha \in \Sigma_N : H_N(\alpha) - a_N \geq C\right]$$
$$\leq 2^N \mathbb{P}\left[H_N(1) - a_N \geq C\right] \leq \text{const} \times \mathrm{e}^{-m_\star C}$$

for large enough N, which can be made arbitrarily small by choosing C large enough. So, it remains to prove that we can find C such that

$$\mathbb{P}\left[\sum_\alpha \mathrm{e}^{H_N(\alpha)-a_N} 1_{H_N(\alpha)-a_N\leq -C} \geq \varepsilon\right] \leq \frac{\varepsilon}{2}.$$

To see the last inequality, remark that the left hand side is bounded by

$$\frac{2^N}{\varepsilon}\mathbb{E}\left[\mathrm{e}^{H_N(1)-a_N} 1_{H_N(\alpha)-a_N\leq -C}\right].$$

We proceed along the lines of Lemma 4.3:

$$\begin{aligned}
&2^N \mathbb{E}\left(\mathrm{e}^{H_N(1)-a_N} 1_{H_N(1)-a_N\leq -C}\right)\\
&= 2^N \mathrm{e}^{N\log E[\exp m\phi] - NE_G(\phi)} \mathrm{e}^{-\omega(N)} \int_{-\infty}^{-C+\omega(N)} \mathrm{e}^{(1-m_\star)y} G_N(dy)\\
&= \mathrm{e}^{-\omega(N)} \int_{-\infty}^{-C+\omega(N)} \mathrm{e}^{(1-m_\star)y} G_N(dy).
\end{aligned}$$

We again use (4.9). Important is now only that G_N, up to a signed measure R_N of total variation $o\left(N^{-1/2}\right)$, is given by a (signed) measure $h_N\left(y/\sqrt{N}\right) dy/\sqrt{N}$, where h_N bounded, uniformly in N. Using that $m_\star < 1$, we get for the smooth part

$$\begin{aligned}
\mathrm{e}^{-\omega(N)} \int_{-\infty}^{-C+\omega(N)} \mathrm{e}^{(1-m_\star)y} \frac{h_N\left(y/\sqrt{N}\right)}{\sqrt{N}} dy &\leq \text{const} \times \frac{\mathrm{e}^{-\omega(N)}}{\sqrt{N}} \int_{-\infty}^{-C+\omega(N)} \mathrm{e}^{(1-m_\star)y} dy\\
&= \text{const} \times \mathrm{e}^{-C(1-m_\star)} \frac{1}{(1-m_\star)\sqrt{N}} \mathrm{e}^{-m_\star \omega(N)}\\
&= \text{const} \times \mathrm{e}^{-C(1-m_\star)} \frac{\sqrt{2\pi v^2}}{1-m_\star}.
\end{aligned}$$

Taking C large, we can make that arbitrarily small. For the R_N-part, we have by partial integration and Fubini:

$$\begin{aligned}
\mathrm{e}^{-\omega(N)} \int_{-\infty}^{-C+\omega(N)} \mathrm{e}^{(1-m_\star)y} R_N(dy) &= \int_{-\infty}^{-C+\omega(N)} R_N(dy) \int_{-\infty}^{(1-m_\star)y} dz\mathrm{e}^{z} \\
&= \mathrm{e}^{-\omega(N)} \int_{-\infty}^{(1-m_\star)(-C+\omega(N))} dz\mathrm{e}^{z} \\
&\times R_N\left(\left(\frac{z}{1-m_\star}, -C+\omega(N)\right]\right) \\
&= o\left(N^{-1/2}\right)\mathrm{e}^{-\omega(N)}\exp\left[(1-m_\star)\left(-C+\omega\left(N\right)\right)\right] \\
&= o\left(1\right).
\end{aligned}$$

■

Proof of Theorem 2.4. We denote by M the space of Radon measures on $(0,\infty)$ endowed with the vague topology. By $\mathcal{H}_N$ we denote the point process associated to the collection of points $(\exp[H_N(\alpha)-a_N], \alpha \in \Sigma_N)$ and $\mathcal{H}$ be its weak limit. We choose a continuous function $h : \mathbb{R}_+ \to \mathbb{R}_+$ with $h(x) = x$ for $x \in [1/C, C]$, $h(x) \leq x$ $\forall x$ and $h(x) = 0$ for $x \notin [1/2C, 2C]$. Then $\int h\, d\mathcal{H}_N$ converges weakly to $\int h\, d\mathcal{H}$ by continuity of the mapping $M \ni \Xi \to \int\, hd\Xi$. By Lemma 4.5, to $\varepsilon > 0$ we can find $C > 0$ large enough such that

$$\mathbb{P}\left[\int_0^{1/C} xd\,\mathcal{H}_N + \int_C^\infty xd\,\mathcal{H}_N \geq \varepsilon\right] \leq \varepsilon,$$

uniformly in N, from which we see by approximation that $\int_0^\infty xd\,\mathcal{H}_N$ converges weakly to $\int_0^\infty xd\,\mathcal{H}$. This also implies that $(\mathcal{H}_N, \int_0^\infty xd\,\mathcal{H}_N)$ converges weakly towards $(\mathcal{H}, \int_0^\infty x\,d\mathcal{H})$. Theorem 2.4 then clearly follows from the continuity of the mapping $M \times (0,\infty) \to M$ defined through $(\Xi, a) \mapsto \Xi\theta_a^{-1}$ with $\theta_a : (0,\infty) \to (0,\infty)$ and $\theta_a(x) \stackrel{\text{def}}{=} x/a$. ■

4.3 The free energy of the REM+Cavity

Proof of Proposition 3.1. Performing the trace over the Ising-spins we have:

$$f_N(\beta) = \frac{1}{N}\log 2^{-N}\sum_\alpha \exp\left[\beta X_\alpha + \sum_i \log\cosh(\beta g_{\alpha,i})\right], \tag{4.10}$$

and we write the X_α's as the sum of N independent standard Gaussians $X_{\alpha,i}$, i.e. $X_\alpha = \sum_{i=1}^N X_{\alpha,i}$. By Theorems 2.2 and 2.3 applies to the function $\phi(x_1,x_2) \stackrel{\text{def}}{=} \beta x_1 + \log\cosh(\beta x_2)$ and $\mu(x_1,x_2)$ a standard, bivariate Gaussian.

The high temperature region $\Gamma'_\phi(1) - \Gamma_\phi(1) \leq \log 2$ is equivalent to

$$E\left[\cosh(\beta g)\log\cosh(\beta g)\right] \leq \mathrm{e}^{\beta^2/2}\log 2,$$

g a standard normal, and E here the expectation with respect to g. So we have to identify this region. We prove that there is a unique $\beta_{\mathrm{cr}} > 0$ such that this inequality holds if and only if $\beta \leq \beta_{\mathrm{cr}}$. To prove that, consider

$$H(\beta) \stackrel{\mathrm{def}}{=} \mathrm{e}^{-\beta^2/2} E[\log\cosh(\beta g)\cosh(\beta g)].$$

Then

$$\begin{aligned}\frac{dH}{d\beta} &= -\beta \mathrm{e}^{-\beta^2/2} E[\cosh(\beta g)\log\cosh(\beta g)] + \mathrm{e}^{-\beta^2/2} E[g\sinh(\beta g)] \\ &+ \mathrm{e}^{-\beta^2/2} E[g\log\cosh(\beta g)\sinh(\beta g)] \\ &= \beta \mathrm{e}^{-\beta^2/2}\left\{E[\cosh(\beta g)] + E[\sinh^2(\beta g)\cosh(\beta g)^{-1}]\right\},\end{aligned}$$

by Gaussian partial integration. So the derivative of H is positive. It is easy to see that $\lim_{\beta\to\infty} H(\beta) = \infty$. So it follows that there is a unique $\beta_{\mathrm{cr}} > 0$ such that $H(\beta_{\mathrm{cr}}) = \log 2$, and $H(\beta) \leq \log 2$ if and only if $\beta \leq \beta_{\mathrm{cr}}$.

If $\beta \leq \beta_{\mathrm{cr}}$, then $f(\beta) = \log E_\mu \mathrm{e}^{m\phi} = \beta^2/2$. If $\beta > \beta_{\mathrm{cr}}$, then we have to determine m_* according to (2.7) which gives (3.5), and plug it into (2.8) which gives the expression for the free energy (3.3). ■

4.4 The Gibbs measure of the REM+Cavity

Proof of Proposition 3.2. Performing the trace over the Ising spins, the Gibbs weight of the pure state $\alpha \in \{1, \ldots, 2^N\}$ in the REM+Cavity reads

$$\mathcal{G}^{(1)}_{\beta,N}(\alpha) = \exp\left[\beta X_\alpha + \sum_{i=1}^N \log\cosh(\beta g_{\alpha,i})\right] \Bigg/ \sum_{\alpha'} \exp\left[\beta X_{\alpha'} + \sum_{i=1}^N \log\cosh(\beta g_{\alpha',i}).\right]$$

As in the proof of the Proposition 3.1, we may then replace the X_α with $\sum_{i=1}^N X_{\alpha,i}$ for a double sequence of independent standard Gaussians $X_{\alpha,i}$, and then we apply Theorem 2.4. ■

For the proof of the other results, we need some remarkable properties of the point processes $\mathrm{PP}(m)$.

Lemma 4.6
Assume that $\{v_\alpha\}_{\alpha\in\mathbb{N}}$ are the points of a $\mathrm{PP}(m)$. Consider also, independently of this point process, a sequence $\{(U_\alpha, V_\alpha)\}_{\alpha\in\mathbb{N}}$ of i.i.d. two dimensional square integrable random vectors satisfying $V_\alpha \geq 1$. Then the following formulas hold:

$$E\left[\frac{\sum_\alpha v_\alpha U_\alpha}{\sum_\alpha v_\alpha V_\alpha}\right] = \frac{E\left[UV^{m_\star-1}\right]}{E\left[V^{m_\star}\right]}, \tag{4.11}$$

$$E\left[\frac{\sum_{\alpha\neq\beta} v_\alpha v_\beta U_\alpha U_\beta}{\left(\sum_\alpha v_\alpha V_\alpha\right)^2}\right] = m_\star\left(\frac{E\left[UV^{m_\star-1}\right]}{E\left[V^{m_\star}\right]}\right)^2, \tag{4.12}$$

$$E\left[\frac{\sum_\alpha v_\alpha^2 U_\alpha^2}{(\sum_\alpha v_\alpha V_\alpha)^2}\right] = (1-m_\star)\frac{E\left[U^2 V^{m_\star-2}\right]}{E\left[V^{m_\star}\right]}. \tag{4.13}$$

For a proof, see [12, Theorem 6.4.5].

Proof of Proposition 3.3. We fix some notation. Let

$$w_\alpha^{(1,2)} = \exp\left(\beta X_\alpha + \sum_{i=3}^N \log\cosh(\beta g_{\alpha,i}) - a_N\right),$$

stand for the (not normalized) Boltzmann weight of the pure state α with a cavity in the sites $i = 1, 2$, and the "centering constant" being given by (4.7) specialized to the present setting. Remark that these weights somewhat differ from the original ones without the cavity, but we clearly still have, in total analogy with Proposition 4.4, weak convergence of $(w_\alpha^{(1,2)})$ towards a collection (v_α) distributed according to a PP (m_*).

For the proof of claim (3.9), expanding the quadratic terms, by symmetry and obvious bounds we have:

$$\begin{aligned}\mathbb{E}\left[\left\langle \delta_{\alpha=\alpha'}\left(q(\sigma,\sigma')-q_\star\right)^2\right\rangle_{\beta,N}^{\otimes 2}\right] &\sim \mathbb{E}\left[\left\langle \delta_{\alpha=\alpha'}\sigma_1\sigma_2\sigma_1'\sigma_2'\right\rangle_{\beta,N}^{\otimes 2}\right] \\ &\quad - 2q_\star\mathbb{E}\left[\left\langle \delta_{\alpha=\alpha'}\sigma_1\sigma_1'\right\rangle_{\beta,N}^{\otimes 2}\right] + q_\star^2\mathbb{E}\left\langle \delta_{\alpha,\alpha'}\right\rangle_{\beta,N}^{\otimes} \\ &= A_1 - A_2 + A_3, \text{ say.}\end{aligned}$$

$\sim$ meaning that the quotient converges to 1, as $N \to \infty$.

The point process $(\mathcal{G}_{\beta,N}^{(1)}(\alpha))_\alpha$ converges to a PD (m_*) which implies

$$A_3 = \mathbb{E}\left[\sum_\alpha \mathcal{G}_{\beta,N}^{(1)}(\alpha)^2\right] \sim 1 - m_\star. \tag{4.14}$$

For A_1, we get

$$\begin{aligned}A_1 &= \mathbb{E}\left[\frac{\sum_\alpha \{\sinh(\beta g_{\alpha,1})\sinh(\beta g_{\alpha,2})\}^2 \left(w_\alpha^{(1,2)}\right)^2}{\left(\sum_\alpha \cosh(\beta g_{\alpha,1})\cosh(\beta g_{\alpha,2}) w_\alpha^{(1,2)}\right)^2}\right] \\ &= \mathbb{E}\left[\frac{\sum_\alpha \{\sinh(\beta g_{\alpha,1})\sinh(\beta g_{\alpha,2})\}^2 \left(\overline{w}_\alpha^{(1,2)}\right)^2}{\left(\sum_\alpha \cosh(\beta g_{\alpha,1})\cosh(\beta g_{\alpha,2}) \overline{w}_\alpha^{(1,2)}\right)^2}\right],\end{aligned}$$

where $\overline{w}_\alpha^{(1,2)} \stackrel{\text{def}}{=} w_\alpha^{(1,2)} \Big/ \sum_\alpha w_\alpha^{(1,2)}$. The corresponding point process converges weakly to PD (m_*), and taking $U_\alpha = \sinh(\beta g_{\alpha,1})\sinh(\beta g_{\alpha,2})$, $V_\alpha = \cosh(\beta g_{\alpha,1})\cosh(\beta g_{\alpha,2})$, a simple domination argument shows that one can pass to the $N \to \infty$ limit, replacing the $\overline{w}_\alpha^{(1,2)}$ by the points of this point process. Applying then (4.13), one gets

$$\begin{aligned}A_1 &\sim (1-m_\star)\frac{E\left[\tanh^2(\beta g_1)\tanh^2(\beta g_2)\cosh(\beta g_1)^{m_\star}\cosh(\beta g_2)^{m_\star}\right]}{E\left[\cosh(\beta g_1)^{m_\star}\cosh(\beta g_2)^{m_\star}\right]} \\ &= (1-m_\star)q_\star^2.\end{aligned}$$

In a similar way, one proves

$$A_2 \sim 2(1-m_\star)q_\star^2.$$

This proves (3.9). (3.10) follows similarly. ∎

Proof of Proposition 3.4. Consider for the moment the β-system only. Denote by $f(\beta)$ the free energy and by $G_{m_\star(\beta)}$ the associated extremal measure solving the GVP. For $\varepsilon > 0$, set $B_{\beta,\varepsilon} \stackrel{\text{def}}{=} B_\varepsilon(G_{m_\star(\beta)}) \subset \mathcal{M}_1^+(\mathbb{R}^2)$ for the open ball of radius ε and center $G_{m_\star(\beta)}$. For $\alpha \in \Sigma_N$ we denote by $L_{N,\alpha}$ the empirical measures associated to the free energies of the pure states.

We first claim that given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\mathbb{P}\left[\mathcal{G}_{\beta,N}\left(\alpha \in \Sigma_N : L_{N,\alpha} \notin B_{\beta,\varepsilon}\right) \geq \mathrm{e}^{-\delta N}\right] \leq \mathrm{e}^{-\delta N}. \tag{4.15}$$

To see this, first observe that uniqueness of the maximizers solving the GVP implies that, with $\phi(x_1, x_2) = \beta x_1 + \log\cosh(\beta x_2)$ and μ standard bivariate Gaussian,

$$f(\beta,\varepsilon) \stackrel{\text{def}}{=} \sup\left\{E_\nu[\phi] - H(\nu \mid \mu) : H(\nu \mid \mu) \leq \log 2, \nu \notin B_{\beta,\varepsilon}\right\} < f(\beta). \tag{4.16}$$

Using the same argument as in the proof of Theorem 2.1 we get

$$\lim_{N\to\infty} \frac{1}{N} \log 2^{-N} \sum_{\alpha : L_{N,\alpha} \notin B_{\beta,\varepsilon}} \mathrm{e}^{\beta X_\alpha + \sum_{i=1}^N \log\cosh(\beta g_{\alpha,i})} \leq f(\beta,\varepsilon),\ \mathbb{P}-a.s.$$

Using the variance estimate from (4.3) and the Tchebychev inequality, it is easily seen that

$$\mathbb{P}[\Lambda_N^c(\beta,\varepsilon)] \leq \exp(-\delta N), \tag{4.17}$$

for some $\delta > 0$, where

$$\Lambda_N(\beta,\varepsilon) = \Bigg\{ 2^{-N} \sum_{\alpha : L_{N,\alpha} \notin B_{\beta,\varepsilon}} \mathrm{e}^{\beta X_\alpha + \sum_i \log\cosh(\beta g_{\alpha,i})} \leq f(\beta,\varepsilon) + \frac{\eta}{3},$$
$$2^{-N} \sum_\alpha \mathrm{e}^{\beta X_\alpha + \sum_i \log\cosh(\beta g_{\alpha,i})} \geq f(\beta,\varepsilon) + \frac{2}{3}\eta \Bigg\}$$

where $\eta \stackrel{\text{def}}{=} f(\beta) - f(\beta,\varepsilon)$. This clearly implies (4.15).

If $\beta \neq \beta'$, then we can choose $\varepsilon > 0$ such that $B_{\beta,\varepsilon} \cap B_{\beta',\varepsilon} = \emptyset$, and the claim a) follows.

As for claim (3.12), we observe that

$$\mathbb{E}\left\langle \delta_{\alpha\neq\alpha'} q(\sigma,\sigma')^2 \right\rangle_{\beta,\beta',N}^{\otimes 2} = \mathbb{E}\left\langle q(\sigma,\sigma')^2 \right\rangle_{\beta,\beta',N}^{\otimes 2} - \mathbb{E}\left\langle \delta_{\alpha=\alpha'} q(\sigma,\sigma')^2 \right\rangle_{\beta,\beta',N}^{\otimes 2}. \tag{4.18}$$

Since $q(\sigma,\sigma')^2 \leq 1$ for all σ,σ', by claim a) of this Proposition the second term on the right hand side in (4.18) is in the limit $N \to \infty$ vanishing. As for the first term on the right hand side of (4.18), by symmetry and obvious bounds we have

$$\mathbb{E}\left\langle q(\sigma,\sigma')^2 \right\rangle_{\beta,\beta',N}^{\otimes 2} = \{1 + O(1/N)\}\, \mathbb{E}\left\langle \sigma_1\sigma_1'\sigma_2\sigma_2' \right\rangle_{\beta,\beta',N}^{\otimes 2} + O(1/N). \tag{4.19}$$

Let us now set $w_\alpha^{(1,2,\beta)} \stackrel{\text{def}}{=} \exp\left[\beta X_\alpha + \sum_{i=3}^N \log\cosh(\beta g_{\alpha,i}) - a_N(\beta)\right]$ for the Boltzmann weight of the pure state α with a cavity in the sites $i = 1, 2$ associated to the β-system, and $a_N(\beta)$ being the centering constant from (4.7) specialized to the setting. Analogously, we write $w_\alpha^{(1,2,\beta')} \stackrel{\text{def}}{=} \exp\left[\beta' X_\alpha + \sum_{i=3}^N \log\cosh(\beta' g_{\alpha,i}) - a_N(\beta')\right]$ in case of the β'-system. With this notations in mind, we write the expectation on the right hand side of (4.18) as

$$
\begin{aligned}
&\mathbb{E}\left\langle \sigma_1\sigma_1'\sigma_2\sigma_2'\right\rangle_{\beta,\beta',N}^{\otimes 2} = \\
&= \mathbb{E}\left[\frac{\sum_\alpha \sinh(\beta g_{\alpha,1})\sinh(\beta g_{\alpha,2}) w_\alpha^{(1,2,\beta)}}{\sum_\alpha \cosh(\beta g_{\alpha,1})\cosh(\beta g_{\alpha,2}) w_\alpha^{(1,2,\beta)}} \times \frac{\sum_{\alpha'} \sinh(\beta' g_{\alpha',1})\sinh(\beta' g_{\alpha',2}) w_{\alpha'}^{(1,2,\beta')}}{\sum_{\alpha'} \cosh(\beta' g_{\alpha',1})\cosh(\beta' g_{\alpha',2}) w_{\alpha'}^{(1,2,\beta')}}\right].
\end{aligned}
$$

By Proposition 4.4.b) the Point Process associated to the collection of "points" of the β-system $(w_\alpha^{(1,2,\beta)})$ converges weakly to a PP $(m_\star(\beta))$, while the Point Process associated to the β'-system converges to a PP $(m_\star(\beta'))$. On the other hand, using similar arguments as in the proof of claim a) it is not difficult to see that the limiting point processes are in fact independent. [Given a compact set $K \subset \mathbb{R}_+$, the $\mathbb{P}$-probability to find a configuration $\alpha \in \Sigma_N$ such that $w_\alpha^{(1,2,\beta)} \in K$ and simultaneously $w_\alpha^{(1,2,\beta')} \in K$ is exponentially small in N.] Hence, the right hand side of (4.19) converges with $N \to \infty$ to the product

$$
E\left[\frac{\sum_\alpha \sinh(\beta g_{\alpha,1})\sinh(\beta g_{\alpha,2}) w_\alpha}{\sum_\alpha \cosh(\beta g_{\alpha,1})\cosh(\beta g_{\alpha,2}) w_\alpha}\right] \times E\left[\frac{\sum_\alpha \sinh(\beta' g_{\alpha,1})\sinh(\beta' g_{\alpha,2}) w_\alpha'}{\sum_\alpha \cosh(\beta' g_{\alpha,1})\cosh(\beta' g_{\alpha,2}) w_\alpha'}\right],
$$

with (w_α) a PP $(m_\star(\beta))$, and (w_α') of a PP $(m_\star(\beta'))$. By (4.11) both expectations are seen to be equal to zero. This settles claim (3.12) of Proposition 3.4. ■

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